MEAN VALUE THEOREMS

Rolle'sTheorem : If a function $f : [a, b] \to R$ is such that

i) It is continuous on $[a, b]$

ii) It is derivable on (a, b) and

iii) $f(a) = f(b)$ then there exists at least one $c \in (a,b)$ such that $f'(c) = 0$.

Lagrange's mean -value theorem or first mean - value theorem :

If a function $f : [a, b] \to R$ is such that

i) It is continuous on $[a, b]$.

ii)It is derivable on (a, b) then there exists at least one $c \in (a,b)$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

VSAQ'S

1.**Verify Rolle's theorem for the function $x^2 - 1$ on $[-1, 1]$.**

Sol. Let $f(x) = x^2 - 1$

f is continuous on $[-1, 1]$ and f is differentiable on $(-1, 1)$

since $f(-1) = f(1) = 0$ and

$\therefore$ By Rolle's theorem $\exists c \in (-1,1)$

Such that $f'(c) = 0$

$f'(x) = 2x = 0$
$\therefore = f'(c) = 0$
$2c = 0 \Rightarrow c = 0$

The point $c = 0 \in (-1, 1)$

Then Rolle's theorem is verified.

2. **$\sin x - \sin 2x$ on $[0, \pi]$.**

Sol. Let $f(x) = \sin x - \sin 2x$

f is continuous on $[0, \pi]$ f is differentiable on $(0, \pi)$

since $f(0) = f(\pi) = 0$ and

$\therefore$ By Rolle's theorem $\exists c \in (0, \pi)$

Such that $f'(c) = 0$

$f'(x) = \cos x - 2\cos 2x$

$f'(c) = 0 \Rightarrow \cos c - 2\cos 2x = 0$

$\Rightarrow \cos c - 2(2\cos^2 c - 1) = 0$

$\cos c - 4\cos^2 c + 2 = 0$

$4\cos^2 c - \cos c - 2 = 0$

$$\cos c = \frac{1 \pm \sqrt{1+32}}{8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\therefore c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$$

3. $\log(x^2 + 2) - \log 3$ on [–1, 1].

Sol. Let $f(x) = \log(x^2 + 2) - \log 3$

f is continuous on [–1, 1] and f is differentiable on (–1, 1)

since $f(-1) = f(1) = 0$ $\therefore$ By Rolle's theorem $\exists c \in (-1, 1)$

Such that $f'(c) = 0$

$$f'(x) = \frac{1}{x^2 + 2}(2x)$$

$$f'(c) = \frac{2c}{c^2 + 2} = 0$$

$2c = 0 \Rightarrow c = 0$

The point $c = 0 \in (-1, 1)$

4. It is given that Rolle's theorem holds for the function $f(x) = x^3 + bx^2 + ax$ on [1, 3] with $c = 2t + \frac{1}{\sqrt{3}}$. Find the values of a and b.

Sol. Given $f(x) = x^3 + bx^2 + ax$

$$f'(x) = 3x^2 + 2bx + a$$

$$\therefore f'(x) = 0 \Leftrightarrow 3c^2 + 2bc + a = 0$$

$$\Leftrightarrow c = \frac{-2b \pm \sqrt{4b^2 - 12a}}{6}$$

$$c = \frac{-b \pm \sqrt{b^2 - 3a}}{3}$$

$$2 + \frac{1}{\sqrt{3}} = \frac{-b \pm \sqrt{b^2 - 3a}}{3}$$

$$\frac{-b}{3} = 2 \text{ and } \frac{\sqrt{b^2 - 3a}}{3} = \frac{1}{\sqrt{3}}$$

$$\Leftrightarrow b = 6 \text{ and } b^2 - 3a = 3$$

$$\Rightarrow 36 - 3 = 3a \Rightarrow 33 = 3a \Rightarrow a = 11$$

Hence a = 11, b = –6.

5. Find a point on the graph of the curve $y = x^3$, where the tangent is parallel to the chord joining (1, 1) and (3, 27).

Sol. Given Points (1, 1) and (3, 27).

Slope of chord = $\frac{27-1}{3-1} = 13$

Given $y = x^3$

$$\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \text{Slope} = 3x^2$$

$$13 = 3x^2 \Rightarrow x^2 = \frac{13}{3}$$

$$\Rightarrow x = \sqrt{\frac{13}{3}} = \frac{\sqrt{39}}{3}$$

$$y = x^3 = \left(\frac{\sqrt{39}}{3}\right)^3 = \frac{39\sqrt{39}}{27} = \frac{13\sqrt{39}}{9}$$

$\therefore$ The point on the curve is $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$

6. Find 'c', so that $f'(c) = \frac{f(b) - f(a)}{b - a}$ in the following case. $f(x) = x^2 - 3x - 1$, a = –11/7, b = 13/7.

Sol. $f(b) = f\left(\frac{13}{7}\right) = \frac{169}{49} - \frac{3(13)}{7} - 1$

$$\frac{169-273-49}{49}=\frac{-153}{49}$$

$$f(a)=f\left(\frac{-11}{7}\right)=\frac{121}{49}-\frac{3(-11)}{7}-1$$

$$=\frac{121+231-49}{49}=\frac{303}{49}$$

$$f'(x)=2x-3$$

$$f'(c)=2c-3$$

Given $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$$2c-3=\frac{\frac{-153}{49}-\frac{303}{49}}{\frac{13}{7}+\frac{11}{7}}=\frac{\frac{-456}{49}}{\frac{24}{7}}$$

$$2c-3=\frac{-456}{49}\times\frac{7}{24}=\frac{-19}{7}$$

$$2c=\frac{-19}{7}+3=\frac{2}{7}\Rightarrow c=\frac{1}{7}$$

7. Verify the Rolle's theorem for the function $(x^2-1)(x-2)$ on $[-1, 2]$. Find the point in the interval where the derivate vanishes.

Sol. Let $f(x) = (x^2 - 1)(x - 2) = x^3 - 2x^2 - x + 2$

f is continuous on $[-1, 2]$

since $f(-1) = f(2) = 0$ and

f is differentiable on $(-1, 2)$

$\therefore$ By Rolle's theorem $\exists c \in (-1, 2)$

Let $f'(c) = 0$

$f'(x) = 3x^2 - 4x - 1$

$3c^2 - 4c - 1 = 0$

$$c=\frac{4\pm\sqrt{16+12}}{6}=\frac{4\pm\sqrt{28}}{6}$$

$$\Rightarrow c=\frac{2\pm\sqrt{7}}{3}$$

8. Verify the conditions of the Lagrange's mean value theorem for the following function. In each case find a point 'c' in the interval as stated by the theorem.

$\sin x - \sin 2x$ on $[0, \pi]$.

Sol. Let $f(x) = \sin x - \sin 2x$

f is continuous on $[0, \pi]$ and

f is differentiable on $(0,\pi)$

Given $f(x) = \sin x - \sin 2x$

$f'(x) = \cos x - 2\cos 2x$

By Lagrange's mean value than $\exists c \in (0,\pi)$

such there

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} \Rightarrow \cos c - 2\cos 2c = 0$$

$$\Rightarrow \cos c - 2(2\cos^2 c - 1) = 0 \Rightarrow \cos c - 4\cos^2 c + 2 = 0$$

$$\Rightarrow 4\cos^2 c - \cos c - 2 = 0 \Rightarrow \cos c = \frac{1 \pm \sqrt{1+32}}{8} = \frac{1 \pm \sqrt{33}}{8}$$

$$\Rightarrow c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$$